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Pairing interaction and its q-deformed versions

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Abstract. We investigate the behaviour of the pairing model within the context of quantum algebras. The pairing Hamiltonian is diagonalized exactly for different values of the deformation parameter q in systems with 8 and 50 particles. The same Hamiltonian is solved with the help of the coherent state variational method. We find that the variational method gives reliable results when compared with the exact ones. We also discuss the differences between the two deformation procedures.

1. Introduction

Recently, the study of *q-deformed* models has received much attention in the literature. Investigations are made either from the mathematical point of view [1] or concerning possible applications to physical systems [2]. The final aim of these works consists of finding a physical meaning to the deformation procedure and, in this way, showing the range of validity and applicability of these models in physics.

Some toy models have already been investigated within the context of quantum algebras. Examples of such studies are the effects of the deformation parameter on the phase transition from the vibrational to the rotational regime in the $su(2)$ Lipkin model [3], in the $su(2) \otimes su(2)$ Moszkowski model [4] and in two superconductivity models, namely the $su(2) \otimes su(2)$ Hubbard and the $su(2)$ Thouless model [5]. For a fixed number of particles in systems described by the models above, it was shown that the phase transition may occur more rapidly, i.e. for weaker interaction strength or even be suppressed, depending on the deformation taken.

In this paper, we consider the pairing interaction in order to gain a better understanding of the effects of the deformation procedure. The variational method is used to calculate the ground-state levels in systems with different numbers of particles and the results are compared with the ones obtained from the exact calculation. One of the reasons for performing the variational calculation in the pairing model is to establish its validity in systems which are more complicated that the one considered in [3], where the method was introduced, and hence justify its applications in systems where the exact calculations are too difficult or even impossible. Even when the exact calculation is possible, the utilization of the variational method can be very useful, since it allows a more natural understanding of critical behaviour in pseudo-spin systems [6].

Quantum deformation is first introduced in a way we refer to as *naive* and the calculations are repeated. We find that the variational method gives reasonably good results when compared with the exact ones. It is also a better approximation for systems with larger numbers of particles. The phase transition of the model is analysed for different

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values of the deformation parameter. It is worth noting that there has been great ambiguity in the procedures used for deforming physical systems because there are various different ways to proceed and all of them go back to the same original system when $q \to 1$. To get rid of this problem, we could demand, as a deformation rule, that all underlying symmetries in the original system be maintained in the deformed system for the quantum algebra [7]. Following the rule we have just suggested, we finally study a second way of introducing deformation and compare the results obtained with the ones we find with the naive deformation.

2. The pairing models

The pairing model [8] consists of two N-fold degenerate levels, whose energy difference is ϵ. The lower level has energy $-\epsilon/2$ and its single-particle states are usually labelled $j_1 m_1$ and the upper level has energy $\epsilon/2$ and its single-particle states are labelled $j_2 m_2$. The pairing Hamiltonian reads

$$H = \frac{\epsilon}{2}\sum_{m}(a^{\dagger}_{j_2 m}a_{j_2 m} - a^{\dagger}_{j_1 m}a_{j_1 m}) - G\sum_{j}\sum_{m>0} a^{\dagger}_{jm}a^{\dagger}_{j\bar{m}}\sum_{j'}\sum_{m'>0} a_{j'\bar{m}'}a_{j'm'} \tag{1}$$

where $a^{\dagger}_{j\bar{m}} = (-1)^{j-m}a_{j-m}$. In what follows, the number of particles (which are fermions), N, will be even and $2j = N/2$.

Introducing the quasi-spin $su(2)$ generators

$$\begin{aligned} S_+ = S_-^{\dagger} &= \sum_{m_1>0} a^{\dagger}_{j_1 m_1}a^{\dagger}_{j_1\bar{m}_1} \\ S_z &= \tfrac{1}{2}\sum_{m_1} a^{\dagger}_{j_1 m_1}a_{j_1 m_1} - \tfrac{1}{4}N \\ L_+ = L_-^{\dagger} &= \sum_{m_2>0} a^{\dagger}_{j_2 m_2}a^{\dagger}_{j_2\bar{m}_2} \\ L_z &= \tfrac{1}{2}\sum_{m_2} a^{\dagger}_{j_2 m_2}a_{j_2 m_2} - \tfrac{1}{4}N \end{aligned} \tag{2}$$

where

$$\begin{aligned} &[S_+, S_-] = 2S_z \qquad && [S_z, S_\pm] = \pm S_\pm \\ &[L_+, L_-] = 2L_z \qquad && [L_z, L_\pm] = \pm L_\pm \\ &[L_i, S_j] = 0 && \end{aligned} \tag{3}$$

one sees that the pairing interaction has an underlying $su(2) \otimes su(2)$ algebra. With the help of (2), equation (1) can be rewritten as

$$H = \epsilon(L_z - S_z) - G(L_+ + S_+)(L_- + S_-). \tag{4}$$

However, the condition $S_z + L_z = 0$ fixes the number of particles [8], and then we obtain

$$\frac{H}{\epsilon} = 2L_z - \frac{G_{\text{eff}}}{N}(L_+L_- + S_+S_- + L_+S_- + S_+L_-) \tag{5}$$

where $G_{\text{eff}} = NG/\epsilon$. The above Hamiltonian is the one we diagonalize exactly and also utilize to find the ground-state energy of the system through the variational method. The basis of states in which (5) is diagonalized is $|S = \frac{1}{4}NL_z, L = \frac{1}{4}N - L_z\rangle$.

Deformation can be introduced straightforwardly by deforming the $su(2)\otimes su(2)$ algebra. The generators of the $su_q(2)$ algebra obey the following commutation relations

$$[J_+, J_-] = [2J_z] \qquad [J_z, J_\pm] = \pm J_\pm \tag{6}$$

where

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{7}$$

q is the deformation parameter such that when $q \to 1$, $[x] = x$. In our case, J can be either L or S. The matrix elements of the deformed $su_q(2)$ operators in a $|jm\rangle$ deformed basis are easily obtained from

$$\begin{aligned} J_z|jm\rangle &= m|jm\rangle \\ J_\pm|jm\rangle &= \sqrt{[j \mp m][j \pm m + 1]}\,|jm \pm 1\rangle. \end{aligned} \tag{8}$$

In this way, one can obtain the eigenvalues of the deformed Hamiltonian for any desired value of the deformation parameter. In this work we are mainly concerned with the ground state and first excited levels. For diagonalizing the deformed Hamiltonian exactly, we use the definition $G_{\text{eff}} = [N]G/\epsilon$ [9].

3. The coherent state variational method

In this section we discuss the variational method used to obtain the ground-state energy where the trial state is given by the $su_q(2)$ coherent state [10]. For this purpose we need the $su_q(2)$ operators in the Bargmann space [10]

$$\begin{aligned} \langle z|J_z|\psi\rangle &= \left(z\frac{\partial}{\partial_z} - j\right)\langle z|\psi\rangle \\ \langle z|J_+|\psi\rangle &= (-q^{-2j}z^2 D_z + [2j]zL_{q^{-1}})\langle z|\psi\rangle \\ \langle z|J_-|\psi\rangle &= D_z\langle z|\psi\rangle \end{aligned} \tag{9}$$

where $|\psi\rangle$ is an arbitrary state and

$$D_z f(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z} \tag{10}$$

is the q-derivative and

$$L_{q^{-1}} f(z) = f(q^{-1}z). \tag{11}$$

When $q \to 1$, the usual formulae are recovered. Finally, we need to calculate

$$E_0 = \min_z \left(\frac{\langle z|H/\epsilon|z\rangle}{\langle z|z\rangle}\right) \tag{12}$$

where

$$|z\rangle = |z_1\rangle \otimes |z_2\rangle = |z_1 z_2\rangle = e_q^{\bar{z}_1 L_+} e_q^{\bar{z}_2 S_+} |0\rangle \tag{13}$$

where $2j = N/2$ and the q-exponential is given by

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \tag{14}$$

and $|z\rangle$ is a state belonging to the $su_q(2) \otimes su_q(2)$ space. Notice that the L operators act on the subspace spanned by the $|z_1\rangle$ states and the S operators act on the subspace spanned by the $|z_2\rangle$ states.

To obtain E_0 in (12) from (5) we start with

$$\frac{\langle z|L_z|z\rangle}{\langle z|z\rangle} = z_1 \bar{z}_1 X_j(z_1 \bar{z}_1) - j \tag{15}$$

where

$$X_j(z\bar{z}) = \sum_{k=0}^{2j-1} \frac{1}{q^{2(j-k)-1} + z\bar{z}} \tag{16}$$

the normalization of the coherent state $|z\rangle$ is

$$\langle z|z\rangle = [1(+)z_1\bar{z}_1]^{2j} [1(+)z_2\bar{z}_2]^{2j} \tag{17}$$

where

$$\langle z_1|z_1\rangle = [1(+)z_1\bar{z}_1]^{2j} = \prod_{k=0}^{2j-1} (1 + q^{2k-2j+1} z_1\bar{z}_1) \tag{18}$$

and a similar equation is used for $\langle z_2|z_2\rangle$. Notice that the q-binomial is given by

$$[a(\pm)b]^m = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} a^{m-k} (\pm b)^k \tag{19}$$

where

$$\begin{bmatrix} m \\ k \end{bmatrix} = \frac{[m]!}{[m-k]![k]!}. \tag{20}$$

From (9), we calculate

$$\frac{\langle z|L_+ L_-|z\rangle}{\langle z|z\rangle} = -q^{-2j}[2j][2j-1] z_1^2 \bar{z}_1^2 \frac{[1(+)z_1\bar{z}_1]^{2j-2}}{[1(+)z_1\bar{z}_1]^{2j}} + [2j]^2 z_1 \bar{z}_1 \frac{[1(+)q^{-1} z_1 \bar{z}_1]^{2j-1}}{[1(+)z_1\bar{z}_1]^{2j}} \tag{21}$$

and

$$\frac{\langle z|L_+ S_-|z\rangle}{\langle z|z\rangle} = [2j]^2 z_1 \bar{z}_2 \frac{[1(+)z_1\bar{z}_1]^{2j-1}}{[1(+)z_1\bar{z}_1]^{2j}} \frac{[1(+)z_2\bar{z}_2]^{2j-1}}{[1(+)z_2\bar{z}_2]^{2j}}. \tag{22}$$

Notice that $\langle z|S_+S_-|z\rangle/\langle z|z\rangle$ can be read off from (21) by exchanging the indices 1 with 2 and $\langle z|S_+L_-|z\rangle/\langle z|z\rangle$ can be read off from (22) by the same exchange. At this point, we can write

$$\begin{aligned}\frac{\langle z|H/\epsilon|z\rangle}{\langle z|z\rangle} &= 2(z_1\bar{z}_1X_j(z_1\bar{z}_1) - j)\\ &\quad - \frac{G_{\text{eff}}}{[N]}\bigg[-q^{-2j}[2j][2j-1]\bigg(z_1^2\bar{z}_1^2\frac{[1(+)z_1\bar{z}_1]^{2j-2}}{[1(+)z_1\bar{z}_1]^{2j}} + z_2^2\bar{z}_2^2\frac{[1(+)z_2\bar{z}_2]^{2j-2}}{[1(+)z_2\bar{z}_2]^{2j}}\bigg)\\ &\quad + [2j]^2\bigg(z_1\bar{z}_1\frac{[1(+)q^{-1}z_1\bar{z}_1]^{2j-1}}{[1(+)z_1\bar{z}_1]^{2j}} + z_2\bar{z}_2\frac{[1(+)q^{-1}z_2\bar{z}_2]^{2j-1}}{[1(+)z_2\bar{z}_2]^{2j}}\\ &\quad + (z_1\bar{z}_2 + z_2\bar{z}_1)\frac{[1(+)z_1\bar{z}_1]^{2j-1}}{[1(+)z_1\bar{z}_1]^{2j}}\frac{[1(+)z_2\bar{z}_2]^{2j-1}}{[1(+)z_2\bar{z}_2]^{2j}}\bigg)\bigg]. \end{aligned} \tag{23}$$

For further convenience, we parametrize the complex numbers z_1 and z_2 as follows

$$z_1 = \tan\tfrac{1}{2}\theta e^{i\phi} \qquad z_2 = \tan\tfrac{1}{2}\theta' e^{i\phi'} \tag{24}$$

where $\theta, \theta' \in [0, \pi]$ and $\phi, \phi' \in [0, 2\pi]$ and use the fact that $L_z + S_z = 0$. From this constraint we have

$$\begin{aligned}\langle z|L_z + S_z|z\rangle &= z_1\bar{z}_1\sum_{k=0}^{N/2-1}\frac{1}{q^{N/2-1-2k} + z_1\bar{z}_1}\\ &\quad + z_2\bar{z}_2\sum_{k=0}^{N/2-1}\frac{1}{q^{N/2-1-2k} + z_2\bar{z}_2} - \frac{N}{2} = 0.\end{aligned} \tag{25}$$

After parametrizing the above equation, it becomes

$$\begin{aligned}&\sin^2\tfrac{1}{2}\theta\sum_{k=0}^{N/2-1}\frac{1}{\cos^2\frac{1}{2}\theta q^{N/2-1-2k} + \sin^2\frac{1}{2}\theta}\\ &\quad + \sin^2\tfrac{1}{2}\theta'\sum_{k=0}^{N/2-1}\frac{1}{\cos^2\frac{1}{2}\theta' q^{N/2-1-2k} + \sin^2\frac{1}{2}\theta'} = 0.\end{aligned} \tag{26}$$

Notice that the above equation shows that θ' is a dependent function of θ. Satisfying this constraint and minimizing H/ϵ to obtain E_0 (which becomes a function of just one variable), we finally obtain

$$\begin{aligned}\frac{E_0}{\epsilon} &= -\frac{N}{2} + 2\sin^2\tfrac{1}{2}\theta B(\theta)\\ &\quad - \frac{G_{\text{eff}}}{[N]}\big[-q^{-N/2}[N/2][N/2-1]C(\theta,\theta') + [N/2]^2D(\theta,\theta')\\ &\quad + 2[N/2]^2\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta'\cos(\phi_1 - \phi_2)E(\theta,\theta')\big]\end{aligned} \tag{27}$$

where

$$B(\theta)=\sum_{k=1}^{N/2}\left(\frac{1}{\cos^2\frac{1}{2}\theta q^{N/2+1-2k}+\sin^2\frac{1}{2}\theta}\right)$$

$$C(\theta,\theta')=c(\theta)+c(\theta')$$

$$c(\theta)=\frac{\sin^4\frac{1}{2}\theta}{(\cos^2\frac{1}{2}\theta+q^{-N/2+1}\sin^2\frac{1}{2}\theta)(\cos^2\frac{1}{2}\theta+q^{N/2-1}\sin^2\frac{1}{2}\theta)}$$

$$D(\theta,\theta')=d(\theta)+d(\theta') \tag{28}$$

$$d(\theta)=\frac{\sin^2\frac{1}{2}\theta}{(\cos^2\frac{1}{2}\theta+q^{N/2-1}\sin^2\frac{1}{2}\theta)}$$

$$E(\theta,\theta')=e(\theta)\times e(\theta')$$

$$e(\theta)=(\cos^2\tfrac{1}{2}\theta)\frac{\prod_{k=0}^{N/2-2}(\cos^2\frac{1}{2}\theta+q^{2k-N/2+2}\sin^2\frac{1}{2}\theta)}{\prod_{k=0}^{N/2-1}(\cos^2\frac{1}{2}\theta+q^{2k-N/2+1}\sin^2\frac{1}{2}\theta)}$$

and $c(\theta')$, $d(\theta')$ and $e(\theta')$ are obtained from $c(\theta)$, $d(\theta)$ and $e(\theta)$, respectively, by exchanging θ with θ'.

For the state of minimum energy

$$\frac{\partial E_0}{\partial(\phi_1-\phi_2)}=0 \tag{29}$$

which implies that $\cos(\phi_1-\phi_2)=1$. In our calculation we have always taken $G_{\rm eff}>0$.

4. Symmetric deformed pairing interaction

As already stated in the introduction, there is a certain degree of ambiguity in the way a physical system can be deformed. In the previous section, we have discussed the deformation of the pairing model in a way that seems to be the most natural one. This is the reason we call it the naive deformation procedure. However, the deformed Hamiltonian does not preserve any symmetries of the original Hamiltonian. In this original Hamiltonian, given in (5), the terms which represent a one-body interaction ($\epsilon(L_z-S_z)$) have an underlying $su(2j_1+1)\otimes su(2j_2+1)$ symmetry. Notice that S_z commutes with all $su(2j_1+1)$ operators which are

$$h_i=a^\dagger_{j_1m_i}a_{j_1m_i}$$

and

$$E_{ij}=a^\dagger_{j_1m_i}a_{j_1m_j} \tag{30}$$

where $i\neq j$. The same statement is valid for L_z. One can show [11] with arguments similar to the ones used by Floratos [12] that the naive deformation procedure does not produce one-body terms with an $su_q(2j_1+1)\otimes su_q(2j_2+1)$ symmetry, which would be the quantum-deformed counterpart of the original system. In analogy to Floratos' deformation technique, there is a special recipe to deform certain Hamiltonians in such a way that they keep their original symmetry. Following his recipe, we can write

$$\begin{aligned} H &= \frac{\epsilon}{2}\frac{1}{2\sinh\gamma/2}\left(\sinh(2\gamma L_z+\gamma N/2)+\sinh(2\gamma L_z-\gamma N/2)\right) \\ &= \frac{\epsilon}{2}\frac{\sinh(2\gamma L_z)\cosh(\gamma N/2)}{2\sinh\gamma/2} \end{aligned} \tag{31}$$

for the interaction strength $G_{\text{eff}}=0$ and where $q=e^{\gamma}$. Notice that the above Hamiltonian bears an underlying $su_q(2)\otimes su_q(2)$ symmetry. The Hamiltonian we diagonalize exactly is the one given above plus the interaction term written in (5).

At this point we have to calculate $\langle z_1|\sinh 2\gamma L_z|z_1\rangle/\langle z_1|z_1\rangle$ which we do as follows

$$\begin{aligned} 2\langle z_1|\sinh 2\gamma L_z|z_1\rangle &= q^{2L_z}\langle z_1|z_1\rangle - q^{-2L_z}\langle z_1|z_1\rangle \\ &= q^{-2j}q^{2z_1\mathrm{d}/\mathrm{d}_{z_1}}[1(+)z_1\bar{z}_1]^{2j} - q^{2j}q^{-2z_1\mathrm{d}/\mathrm{d}_{z_1}}[1(+)z_1\bar{z}_1]^{2j}. \end{aligned} \tag{32}$$

Here we need to use the following property

$$q^{\alpha z\mathrm{d}/\mathrm{d}_z}f(z)=f(q^{\alpha}z) \tag{33}$$

where

$$f(z)=\sum_{n=0}^{2j}a_n z^n.$$

Equation (32) can be rewritten as

$$\frac{\langle z_1|\sinh 2\gamma L_z|z_1\rangle}{\langle z_1|z_1\rangle}=\frac{1}{2}\left(q^{-2j}\frac{[1(+)q^2z_1\bar{z}_1]^{2j}}{[1(+)z_1\bar{z}_1]^{2j}}-q^{2j}\frac{[1(+)q^{-2}z_1\bar{z}_1]^{2j}}{[1(+)z_1\bar{z}_1]^{2j}}\right). \tag{34}$$

With the help of (18), we obtain

$$\frac{[1(+)q^2z\bar{z}]^{2j}}{[1(+)z\bar{z}]^{2j}}=\frac{(1+q^{2j+1}z\bar{z})}{(1+q^{-2j+1}z\bar{z})}$$

and

$$\frac{[1(+)q^{-2}z\bar{z}]^{2j}}{[1(+)z\bar{z}]^{2j}}=\frac{(1+q^{-2j-1}z\bar{z})}{(1+q^{2j-1}z\bar{z})}. \tag{35}$$

Substituting the above expressions properly in (34), it becomes

$$\frac{\langle z_1|\sinh 2\gamma L_z|z_1\rangle}{\langle z_1|z_1\rangle}=\frac{[2j](z_1^2\bar{z}_1^2-1)(q-q^{-1})}{2(1+q^{-2j+1}z_1\bar{z}_1)(1+q^{2j-1}z_1\bar{z}_1)} \tag{36}$$

and an analogous expression is obtained for $\langle z_2|\sinh 2\gamma S_z|z_2\rangle/\langle z_2|z_2\rangle$. Finally

$$\begin{aligned} \frac{\langle z|H/\epsilon|z\rangle}{\langle z|z\rangle} &= \frac{[2j](q^{N/2}+q^{-N/2})}{2[1/2]}\frac{(\sin^2\frac{1}{2}\theta-\cos^2\frac{1}{2}\theta)}{(\cos^2\frac{1}{2}\theta+q^{-N/2+1}\sin^2\frac{1}{2}\theta)(\cos^2\frac{1}{2}\theta+q^{N/2-1}\sin^2\frac{1}{2}\theta)} \\ &\quad -\frac{G_{\text{eff}}}{[N]}\Big[-q^{-N/2}[N/2][N/2-1]C(\theta,\theta')+[N/2]^2D(\theta,\theta') \\ &\quad +2[N/2]^2\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta'\cos(\phi_1-\phi_2)E(\theta,\theta')\Big] \end{aligned} \tag{38}$$

where $C(\theta,\theta')$, $D(\theta,\theta')$ and $E(\theta,\theta')$ are given in (28).

5. Results and conclusion

To start with, we have checked the effect of the quantum deformation on the pairing model. For this purpose we have performed an exact calculation to obtain the eigenenergies for systems with 8 and 50 particles. With these results we have observed the thermodynamic phase transition undergone by both systems. In figures 1 and 2 we have plotted $(E_1 - E_0)/\epsilon$ as a function of the interaction strength $G_{\rm eff}$ for different values of q, in order to study the phase transition. It is easily observed that the phase transition is gradually suppressed with increasing q for a fixed number of particles. The same is true if q is kept fixed and the number of particles is increased.

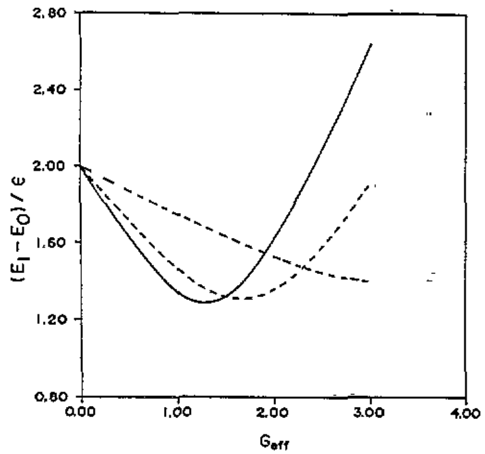

Figure 1. $(E_1 - E_0)/\epsilon$ as a function of the interaction strength $G_{\rm eff}$ obtained from the exact diagonalization of (5) for eight particles. The full curve is drawn for $q = 1.0$, the broken curve for $q = 1.2$ and the long broken curve for $q = 1.5$.

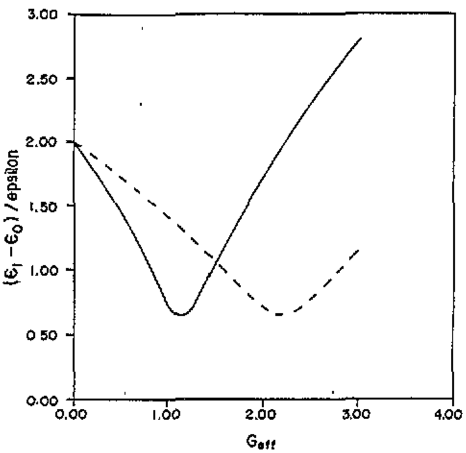

Figure 2. The same as in figure 1, but for 30 particles. The full curve shows the results for $q = 1.0$ and the broken curve for $q = 1.05$.

Secondly, we have studied the validity of the variational method. In doing so, we have deformed the systems in two different ways, called the naive and symmetric deformation procedures. In figures 3 and 4 we show a comparison between the ground-state energies obtained exactly and variationally for a system with eight particles and different values of q. In figure 3, we utilized the naive deformation procedure and in figure 4 the symmetric one. We have done the same in figures 5 and 6 for a much larger system, where 50 particles have been considered. Analysing these four figures, we observe that the variational method is indeed a good approximation even in the case of the pairing model, which is much more complex that the previously studied Lipkin model [3]. The method improves when more particles are taken into account for a fixed deformation and also for lower deformations when the number of particles is fixed. It is worth mentioning that both deformation procedures, i.e. the naive and the symmetric, give the same qualitative behaviour. However, the symmetric deformation procedure seems to enhance the effects of the q-deformation.

We have not plotted phase transition curves obtained from Hamiltonian (31) because it gives a behaviour analogous to the one shown in figures 1 and 2.

Finally, we would like to point out that the variational method is also valid in complex systems even when quantum deformation (either naive or symmetric) is introduced. This fact helps in calculations where the exact result is not easily obtained.

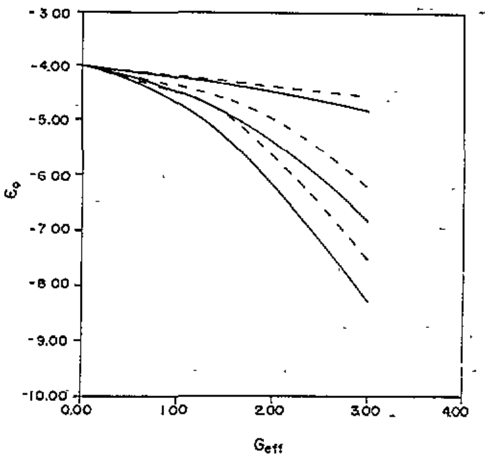

Figure 3. The ground-state energies (E_0) as a function of the interaction strength $G_{\rm eff}$ for eight particles. The full curves show the results obtained from the exact diagonalization of (5) for $q = 1.0$, 1.2 and 1.5 (lower curves represent smaller deformations). The broken curves show the results obtained from the variational method also for $q = 1.0$, 1.2 and 1.5.

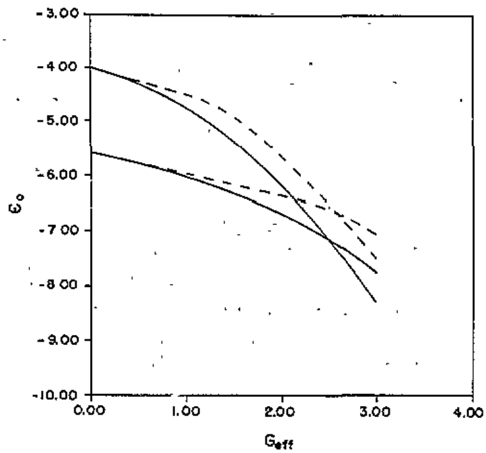

Figure 4. The same as in figure 3, but using the symmetric deformation procedure, i.e. equation (31) plus the interaction terms of equation (5) is exactly diagonalized and also solved with the variational method. The full curves represent the exact results and the broken curves the variational ones. In this case, the lower curves are the ones containing the quantum deformation.

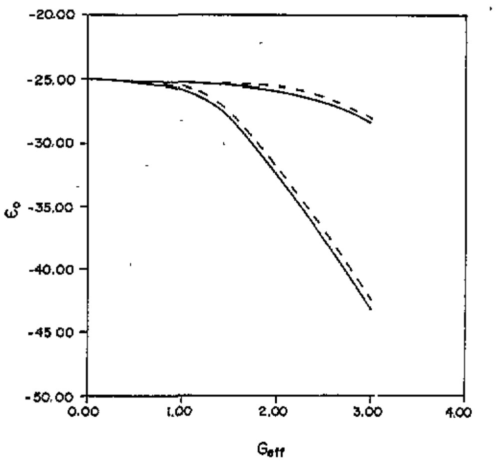

Figure 5. The same as in figure 3, but for 50 particles. The lower curves are plotted for $q = 1.0$ and the upper ones for $q = 1.05$.

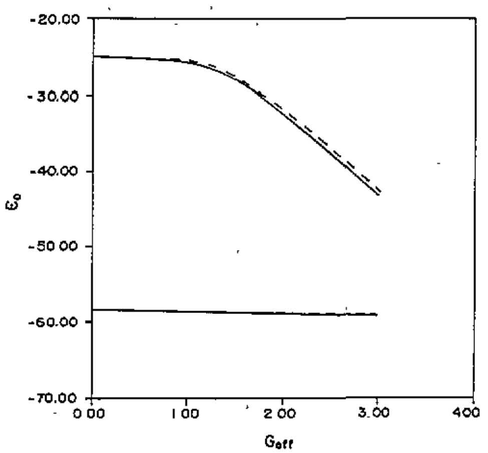

Figure 6. The same as in figure 5, but curves are obtained from the symmetric deformation procedure. The lower curves are the ones obtained with the introduction of $q = 1.05$.

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